Optim Lett (2016) 10:457–472
DOI 10.1007/s11590-014-0784-0

ORIGINAL PAPER

Global solutions to a class of CEC benchmark constrained optimization problems

Xiaojun Zhou · David Yang Gao · Chunhua Yang

Received: 6 March 2013 / Accepted: 16 August 2014 / Published online: 29 August 2014

Abstract This paper aims to solve a class of CEC benchmark constrained optimization problems that have been widely studied by nature-inspired optimization algorithms. Based on canonical duality theory, these challenging problems can be reformulated as a unified canonical dual problem over a convex set, which can be solved deterministically to obtain global optimal solutions in polynomial time. Applications are illustrated by some well-known CEC benchmark problems, and comparisons with other methods have demonstrated the effectiveness of the proposed approach.

Keywords Global optimization · Constrained optimization · Canonical duality theory · CEC benchmark

1 Introduction

Nature-inspired optimization algorithms, such as genetic algorithm (GA), evolution strategy (ES), particle swarm optimization (PSO), differential evolution (DE) and state transition algorithm (STA) [26,27], have received considerable attention in recent decades due to their strong adaptability and easy implementation. Strictly speaking, these algorithms are unconstrained optimization procedures, and therefore it is necessary to find techniques to deal with constraints when solving constrained optimization problems. The most common approach to handling constraints is the penalty function

X. Zhou (✉)· C. Yang
School of Information Science and Engineering, Central South University,
Changsha 410083, China
e-mail: tiezhongyu2010@gmail.com

X. Zhou · D. Y. Gao
School of Science, Information Technology and Engineering, Federation University Australia,
Victoria 3353, Australia

method. The idea of this method is to transform a constrained optimization problem into an unconstrained one by adding a certain term to the objective function based on the amount of constraint violation. Then, some special representations and operators can be designed to preserve the feasibility of solutions at all times or to repair a solution when it is infeasible. Multiobjective optimization techniques are also used to manage constraints. The main idea is to rewrite the single objective optimization problem as a multiobjective optimization problem in which the constraints in the original problem are treated as additional objectives [6,19].

On the other hand, regarding some constrained optimizations with special structures, for instance, the nonconvex quadratically constrained quadratic programs (QCQP), deterministic global optimization techniques are prevalent. By successive linearization within a branching tree using reformulation-linearization techniques (RLT) to estimate all quadratic terms, a branch and cut algorithm for nonconvex QCQP was proposed in [2]. A simplicial branch-and-bound algorithm for QCQP was given in [16], in which, branching is done by partitioning the feasible region into the Cartesian product of two-dimensional triangles and rectangles. Based on the brand-and-bound scheme, through piecewise-linear and edge-concave relaxations, a deterministic global optimization approach was proposed for solving mixed-integer QCQP [20]. Semidefinite and conic relaxations for QCQP are also ubiquitous in recent years, please see [1,3,18] and references therein.

It is known that the traditional Lagrange multiplier method can be used mainly for solving convex optimization problems. If either the objective function or its feasible set is nonconvex, the well-developed Lagrangian duality produces a duality gap in global optimization [4,5,11]. In order to bridge the gap inherent in the classical Lagrange duality theory, a canonical duality theory has been developed during the last decades. This potentially powerful theory was originated in the late 1980s by Gao and Strang [9] from nonconvex mechanics. The kernels of the theory consist of a canonical dual transformation methodology, a complementary-dual principle, and a triality theory. The main merit is that by using this theory, a large class of nonconvex/nonsmoonth/discrete optimization/variational problems in totally different fields can be transformed as a unified canonical dual problem without duality gap, which is a concave maximization over a convex domain. Under certain conditions, a canonical dual problem can be solved easily, by many well-developed algorithms and softwares [12]. However, if there exists no critical point in the canonical dual feasible space, we cannot get the corresponding global solution to the primal problem. In this case, certain linear and nonlinear perturbation methods have been developed to recover the global optimal solutions [8,14,21,22,25]. In [10], the standard quadratic programming (QP) problem with quadratic objective function and linear constraints was studied by the canonical duality. In this paper, we study the quadratic optimization problem with quadratic and box constraints and focus on solving a class of Congress on Evolutionary Computation (CEC) benchmark constrained optimization problems that have been widely studied by nature-inspired algorithms. By integrating the canonical dual solution with the KKT conditions, we are able to obtain approximate or global solutions easily, and experimental results have testified the effectiveness of the proposed approach when compared with other methods.

2 The canonical duality theory

In this paper, we focus on the following quadratic optimization problem with quadratic and box constraints (primal problem):

$$
\begin{aligned}
(\mathcal{P}) : \min \ & \left\{ P(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x} - \boldsymbol{a}^T\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^n \right\}, \\
\text{s.t. } & \boldsymbol{g}(\boldsymbol{x}) = \{g_j(\boldsymbol{x})\} = \left\{ \frac{1}{2}\boldsymbol{x}^T B_j\boldsymbol{x} - \boldsymbol{b}_j^T\boldsymbol{x} - b_j \right\} \le \boldsymbol{0},\ j = 1, \cdots, m, \\
& c_i \le x_i \le d_i,\ i = 1, \cdots, n,
\end{aligned} \tag{1}
$$

where, $\boldsymbol{x} = (x_1, \cdots, x_n)$, $A = A^T$, $B_j = B_j^T \in \mathbb{R}^{n\times n}$ are symmetric matrices, $\boldsymbol{a}, \boldsymbol{b}_j \in \mathbb{R}^n$ are given vectors, b_j, c_i, d_i are constant.

Let $E_i \in \mathbb{R}^{n\times n}$, $\boldsymbol{e}_i \in \mathbb{R}^n$ be a diagonal matrix and a unit vector, with all zeros except a one in the position (i, i) and (i), respectively. Let denote $B_k = 2E_k$, $\boldsymbol{b}_k = (c_k + d_k)\boldsymbol{e}_k$, $b_k = c_k d_k$, $k = m + 1, \cdots, m + n$, then constraints in $(\mathcal{P})$ can be uniformly rewritten as the so-called geometrical operator

$$
\boldsymbol{\xi} = \Lambda(\mathbf{x}) = \left\{ \frac{1}{2}\boldsymbol{x}^T B_k\boldsymbol{x} - \boldsymbol{b}_k^T\boldsymbol{x} - b_k \right\} : \mathbb{R}^n \to \mathcal{E}_a \subset \mathbb{R}^{m+n}, \tag{2}
$$

where $\mathcal{E}_a = \{\boldsymbol{\xi} \in \mathbb{R}^{m+n} |\ \ \boldsymbol{\xi} \le 0 \in \mathbb{R}^{m+n}\}$. Therefore, by introducing an indicator function

$$
V(\boldsymbol{\xi}) = \begin{cases} 0 & \text{if } \boldsymbol{\xi} \in \mathcal{E}_a \\ +\infty & \text{otherwise} \end{cases} \tag{3}
$$

where $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_{m+n})$, and let $U(\boldsymbol{x}) = -P(\boldsymbol{x}) = -\frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x} + \boldsymbol{a}^T\boldsymbol{x}$, the primal problem $(\mathcal{P})$ can be written into the following unconstrained canonical form:

$$
\min\{\Pi(\boldsymbol{x}) = V(\Lambda(\boldsymbol{x})) - U(\boldsymbol{x}) : \boldsymbol{x} \in \mathbb{R}^n\}. \tag{4}
$$

By the Fenchel transformation, the conjugate function $V^\sharp(\boldsymbol{\varsigma})$ of $V(\boldsymbol{\xi})$ can be defined by

$$
V^\sharp(\boldsymbol{\varsigma}) = \sup_{\xi}\{\boldsymbol{\xi}^T\boldsymbol{\varsigma} - V(\boldsymbol{\xi})\} = \begin{cases} 0 & \text{if } \boldsymbol{\varsigma} \in \mathcal{S}_a \\ +\infty & \text{otherwise} \end{cases} \tag{5}
$$

where $\mathcal{S}_a = \{\boldsymbol{\varsigma} \in \mathbb{R}^{m+n} | \boldsymbol{\varsigma} \ge 0\}$. By convex analysis, we have the following canonical duality relations

$$
\boldsymbol{\varsigma} \in \partial V(\boldsymbol{\xi}) \Leftrightarrow \boldsymbol{\xi} \in \partial V^\sharp(\boldsymbol{\varsigma}) \Leftrightarrow V(\boldsymbol{\xi}) + V^\sharp(\boldsymbol{\varsigma}) = \boldsymbol{\xi}^T\boldsymbol{\varsigma},
$$

which are equivalent to the following KKT conditions:

$$
\boldsymbol{\xi} \in \mathcal{E}_a, \quad \boldsymbol{\varsigma} \in \mathcal{S}_a, \quad \boldsymbol{\xi} \perp \boldsymbol{\varsigma}.
$$

Let replace $V(\Lambda(\boldsymbol{x}))$ by $\Lambda^T(\boldsymbol{x})\boldsymbol{\varsigma} - V^\sharp(\boldsymbol{\varsigma})$. The so-called total complementary function $\Xi : \mathbb{R}^n \times \mathcal{S}_a \to \mathbb{R}$ associated with $\Pi(\boldsymbol{x})$ can be defined as

$$\begin{aligned}\Xi(\boldsymbol{x}, \boldsymbol{\varsigma}) &= \Lambda^T(\boldsymbol{x})\boldsymbol{\varsigma} - V^\sharp(\boldsymbol{\varsigma}) - U(\boldsymbol{x}) \\ &= \frac{1}{2}\boldsymbol{x}^T G(\boldsymbol{\varsigma})\boldsymbol{x} - \boldsymbol{x}^T F(\boldsymbol{\varsigma}) - \boldsymbol{\varsigma}^T \boldsymbol{d},\end{aligned} \tag{6}$$

where $\boldsymbol{d} = (b_1, \cdots, b_{m+n})^T$, $\boldsymbol{\varsigma} = (\varsigma_1, \cdots, \varsigma_{m+n}) \in \mathcal{S}_a$, and

$$G(\boldsymbol{\varsigma}) = A + \sum_{k=1}^{m+n} \varsigma_k B_k, \quad F(\boldsymbol{\varsigma}) = \boldsymbol{a} + \sum_{k=1}^{m+n} \varsigma_k \boldsymbol{b}_k.$$

By the fact that $\Xi(\boldsymbol{x}, \boldsymbol{\varsigma})$ is a quadratic function of $\boldsymbol{x}$, the criticality condition $\nabla_{\boldsymbol{x}}\Xi(\boldsymbol{x}, \boldsymbol{\varsigma}) = 0$ leads to a linear equation $G(\boldsymbol{\varsigma})\boldsymbol{x} = F(\boldsymbol{\varsigma})$. Therefore, solving this equation to eliminate $\boldsymbol{x}$ in $\Xi(\boldsymbol{x}, \boldsymbol{\varsigma})$, the canonical dual function can be formulated as

$$P^d(\boldsymbol{\varsigma}) = -\frac{1}{2}F^T(\boldsymbol{\varsigma})G^{-1}(\boldsymbol{\varsigma})F(\boldsymbol{\varsigma}) - \boldsymbol{\varsigma}^T\boldsymbol{d}. \tag{7}$$

Finally, the canonical dual problem can be described as follows:

$$(\mathcal{P}^d): \quad \max\{P^d(\boldsymbol{\varsigma}) |\ \boldsymbol{\varsigma} \in \mathcal{S}_a^+\} \tag{8}$$

where the canonical dual feasible space is defined by

$$\mathcal{S}_a^+ = \{\boldsymbol{\varsigma} \in \mathcal{S}_a |\ G(\boldsymbol{\varsigma}) \succ 0\}.$$

Theorem 1 *The problem $(\mathcal{P}^d)$ is canonically dual to $(\mathcal{P})$ in the sense that if $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\varsigma}})$ is a KKT point of $\Xi(\boldsymbol{x}, \boldsymbol{\varsigma})$, then $\bar{\boldsymbol{x}}$ is a KKT point of $(\mathcal{P})$, $\bar{\boldsymbol{\varsigma}}$ is a KKT point of $(\mathcal{P}^d)$, and*

$$P(\bar{\boldsymbol{x}}) = \Xi(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\varsigma}}) = P^d(\bar{\boldsymbol{\varsigma}}).$$

Moreover, if $\bar{\boldsymbol{\varsigma}} \in \mathcal{S}_a^+$, then $\bar{\boldsymbol{x}} = G^{-1}(\bar{\boldsymbol{\varsigma}})F(\bar{\boldsymbol{\varsigma}})$ is the global minimizer of $(\mathcal{P})$.

Proof By introducing Lagrange multiplier $\boldsymbol{\xi} \in \mathcal{E}_a$ associated with $\boldsymbol{\varsigma} \geq 0$, the Lagrangian $L(\boldsymbol{\xi}, \boldsymbol{\varsigma})$ is given by

$$L(\boldsymbol{\xi}, \boldsymbol{\varsigma}) = -\frac{1}{2}F^T(\boldsymbol{\varsigma})G^{-1}(\boldsymbol{\varsigma})F(\boldsymbol{\varsigma}) - \boldsymbol{\varsigma}^T\boldsymbol{d} - \boldsymbol{\xi}^T\boldsymbol{\varsigma}. \tag{9}$$

It is easy to prove that the criticality conditions $\nabla_{\boldsymbol{\varsigma}} L(\boldsymbol{\xi}, \boldsymbol{\varsigma}) = 0$ lead to

$$\boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \cdots \\ \xi_{m+n} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\bar{\boldsymbol{x}}^T B_1 \bar{\boldsymbol{x}} - \boldsymbol{b}_1^T \bar{\boldsymbol{x}} - b_1 \\ \cdots \\ \frac{1}{2}\bar{\boldsymbol{x}}^T B_{m+n} \bar{\boldsymbol{x}} - \boldsymbol{b}_{m+n}^T \bar{\boldsymbol{x}} - b_{m+n} \end{pmatrix} \tag{10}$$

and the accompanying KKT conditions include

$$0 \le \bar{\varsigma}_k \perp \frac{1}{2}\bar{\boldsymbol{x}}^T B_k \bar{\boldsymbol{x}} - \boldsymbol{b}_k^T \bar{\boldsymbol{x}} - b_k \le 0, k = 1, \ldots, m+n. \tag{11}$$

Therefore, $\bar{\boldsymbol{x}}$ is a KKT point of $(\mathcal{P})$. Furthermore, since $\bar{\boldsymbol{\varsigma}} \ge 0$ for any $\Lambda(\boldsymbol{x}) \le 0$, we have

$$\begin{aligned} P(\boldsymbol{x}) &\ge P(\boldsymbol{x}) + \bar{\boldsymbol{\varsigma}}^T \Lambda(\boldsymbol{x}) \\ &= \frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x} - \boldsymbol{a}^T \boldsymbol{x} + \sum_{k=1}^{m+n} \left(\frac{1}{2}\boldsymbol{x}^T \bar{\varsigma}_k B_k \boldsymbol{x} - \bar{\varsigma}_k \boldsymbol{b}_k^T \boldsymbol{x} - \bar{\varsigma}_k b_k \right) \\ &= \frac{1}{2}\boldsymbol{x}^T G(\bar{\boldsymbol{\varsigma}})\boldsymbol{x} - \boldsymbol{x}^T F(\bar{\boldsymbol{\varsigma}}) - \bar{\boldsymbol{\varsigma}}^T \boldsymbol{d} \\ &= \Xi(\boldsymbol{x}, \bar{\boldsymbol{\varsigma}}). \end{aligned} \tag{12}$$

Noting that $P(\bar{\boldsymbol{x}}) = \Xi(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\varsigma}}), \nabla_{\boldsymbol{x}} \Xi(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\varsigma}}) = 0$ and $\Xi(\boldsymbol{x}, \bar{\boldsymbol{\varsigma}})$ is a quadratic function with respect to $\boldsymbol{x}$, we have

$$\begin{aligned} P(\boldsymbol{x}) - P(\bar{\boldsymbol{x}}) &\ge \Xi(\boldsymbol{x}, \bar{\boldsymbol{\varsigma}}) - \Xi(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\varsigma}}) \\ &= (\boldsymbol{x} - \bar{\boldsymbol{x}})\nabla_{\boldsymbol{x}} \Xi(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\varsigma}}) + \frac{1}{2}(\boldsymbol{x} - \bar{\boldsymbol{x}})^T \nabla_{\boldsymbol{xx}} \Xi(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\varsigma}})(\boldsymbol{x} - \bar{\boldsymbol{x}}) \\ &= \frac{1}{2}(\boldsymbol{x} - \bar{\boldsymbol{x}})^T G(\bar{\boldsymbol{\varsigma}})(\boldsymbol{x} - \bar{\boldsymbol{x}}). \end{aligned} \tag{13}$$

If $G(\bar{\boldsymbol{\varsigma}}) \succ 0$, it is easy to find that $\bar{\boldsymbol{x}}$ is the global minimizer of $(\mathcal{P})$, where, $\bar{\boldsymbol{x}}$ is a solution of the canonical equilibrium equation

$$G(\bar{\boldsymbol{\varsigma}})\bar{\boldsymbol{x}} = F(\bar{\boldsymbol{\varsigma}}). \tag{14}$$

3 Implementation techniques

By Theorem 1 we know that the canonical dual problem $(\mathcal{P}^d)$ is a concave maximization over a convex set, which can be solved by well-developed nonlinear optimization techniques. In the section, we show that $(\mathcal{P}^d)$ can be relaxed to a semi-definite programming (SDP) problem. Therefore, the popular software can be used to solve some benchmark problems. First, we rewrite $(\mathcal{P}^d)$ into the following relaxed form:

$$\begin{aligned} \min \quad & g + \boldsymbol{\varsigma}^T \boldsymbol{d} \\ \text{s.t.} \quad & g \ge \frac{1}{2} F^T(\boldsymbol{\varsigma}) G^{-1}(\boldsymbol{\varsigma}) F(\boldsymbol{\varsigma}) && (15) \\ & G(\boldsymbol{\varsigma}) \succeq 0 && (16) \\ & \boldsymbol{\varsigma} \ge 0 && (17) \end{aligned}$$

where g is actually the pure gap function in the canonical duality theory (see [13]). Using the Schur complement [7], we can get the equivalent positive (semi) definite condition to (15) and (16)

$$\begin{pmatrix} G(\boldsymbol{\varsigma}) & F(\boldsymbol{\varsigma}) \\ F^T(\boldsymbol{\varsigma}) & 2g \end{pmatrix} \succeq 0 \tag{18}$$

and then the optimization problem can be expressed as the standard SDP form

$$\begin{aligned} \min \quad & g + \boldsymbol{\varsigma}^T \boldsymbol{d} \\ \text{s.t.} \quad & \begin{pmatrix} G(\boldsymbol{\varsigma}) & F(\boldsymbol{\varsigma}) \\ F^T(\boldsymbol{\varsigma}) & 2g \end{pmatrix} \succeq 0 \qquad (19) \\ & \boldsymbol{\varsigma} \geq 0 \qquad (20) \end{aligned}$$

If $G(\bar{\boldsymbol{\varsigma}}) \succ 0$, we can get the corresponding global solution to $(\mathcal{P})$ by the canonical duality theory. In practice, the estimation of $G(\bar{\boldsymbol{\varsigma}})$ may exist little inaccuracy due to the perturbed complementary slackness in primal-dual interior point method and numerical precision. In this study, we use the Cholesky factorization, condition number and the smallest eigenvalue of $G(\bar{\boldsymbol{\varsigma}})$ to evaluate the positive definiteness comprehensively. If $G(\bar{\boldsymbol{\varsigma}})$ is ill conditioned or $\det(G(\bar{\boldsymbol{\varsigma}})) = 0$, we can add a linear perturbation to the primal objective function and then integrate the canonical dual solutions with the KKT conditions to recover the approximate or global solution to primal problem. Details of the techniques are given in the following examples.

4 Numerical results

Most of the benchmark constrained optimization problems are from [15], and we keep the number of each problem. In the experiments, we use SeDuMi [23] (a software package which can solve SDP problems) to obtain the canonical dual solution. The built-in functions *fsolve* and *fminunc* in MATLAB optimization toolbox are also used to solve the simple nonlinear equations and unconstrained optimization problems.

Example 1 g01

$$\begin{aligned} \min f(\boldsymbol{x}) = 5\sum_{i=1}^{4} x_i - 5\sum_{i=1}^{4} x_i^2 - \sum_{i=5}^{13} x_i & \\ \text{s.t.} \quad g_1(\boldsymbol{x}) &= 2x_1 + 2x_2 + x_{10} + x_{11} - 10 \leq 0 \\ g_2(\boldsymbol{x}) &= 2x_1 + 2x_3 + x_{10} + x_{12} - 10 \leq 0 \\ g_3(\boldsymbol{x}) &= 2x_2 + 2x_3 + x_{11} + x_{12} - 10 \leq 0 \\ g_4(\boldsymbol{x}) &= -8x_1 + x_{10} \leq 0 \\ g_5(\boldsymbol{x}) &= -8x_2 + x_{11} \leq 0 \\ g_6(\boldsymbol{x}) &= -8x_3 + x_{12} \leq 0 \\ g_7(\boldsymbol{x}) &= -2x_4 - x_5 + x_{10} \leq 0 \end{aligned}$$

$$g_8(\boldsymbol{x}) = -2x_6 - x_7 + x_{11} \leq 0$$
$$g_9(\boldsymbol{x}) = -2x_8 - x_9 + x_{12} \leq 0$$

where the bounds are $0 \leq x_i \leq 1 (i = 1, \cdots, 9)$, $0 \leq x_i \leq 100 (i = 10, 11, 12)$ and $0 \leq x_{13} \leq 1$.

Solving the canonical dual problem, we can obtain $\bar{\boldsymbol{\varsigma}} =$

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
\varsigma_1 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 & \varsigma_8 & \varsigma_9 & \varsigma_{10} & \varsigma_{11} \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 1.0000 & 1.0000 & 1.0000 & 5.0000 & 5.0000 \\
\varsigma_{12} & \varsigma_{13} & \varsigma_{14} & \varsigma_{15} & \varsigma_{16} & \varsigma_{17} & \varsigma_{18} & \varsigma_{19} & \varsigma_{20} & \varsigma_{21} & \varsigma_{22} \\
5.0000 & 7.0001 & 2.0001 & 3.0001 & 2.0001 & 3.0001 & 2.0001 & -0.0000 & -0.0000 & -0.0000 & 1.0001
\end{array}\right)$$

In this case, $G(\bar{\boldsymbol{\varsigma}})$ is positive seme-definite but singular, satisfying the global optimality condition. By the KKT conditions, we can find that g_7, g_8, g_9, bounds of $x_1, \cdots, x_9$, and x_{13} are active, so we can first get

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} & x_{13} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & 1
\end{array}\right)$$

where, "$-$" means undetermined. Considering that constraints g_7, g_8, g_9 are active, solving the corresponding linear equations, we can easily get $x_{10} = 3, x_{11} = 3, x_{12} = 3$. Finally, the global solution to g01 is $\boldsymbol{x}^* =$

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c}
x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} & x_{11} & x_{12} & x_{13} \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 3 & 1
\end{array}\right)$$

and $f(\boldsymbol{x}^*) = -15$.

Example 2 g04

$$\begin{aligned}
\min f(\boldsymbol{x}) &= 5.3578547x_3^2 + 0.8356891x_1x_5 + 37.293239x_1 - 40792.141 \\
\text{s.t.} \quad & g_1(\boldsymbol{x}) = 85.334407 + 0.0056858x_2x_5 + 0.0006262x_1x_4 - 0.0022053x_3x_5 - 92 \leq 0 \\
& g_2(\boldsymbol{x}) = -85.334407 - 0.0056858x_2x_5 - 0.0006262x_1x_4 + 0.0022053x_3x_5 \leq 0 \\
& g_3(\boldsymbol{x}) = 80.51249 + 0.0071317x_2x_5 + 0.0029955x_1x_2 + 0.0021813x_3^2 - 110 \leq 0 \\
& g_4(\boldsymbol{x}) = -80.51249 - 0.0071317x_2x_5 - 0.0029955x_1x_2 - 0.0021813x_3^2 + 90 \leq 0 \\
& g_5(\boldsymbol{x}) = 9.30096 + 0.0047026x_3x_5 + 0.0012547x_1x_3 + 0.0019085x_3x_4 - 25 \leq 0 \\
& g_6(\boldsymbol{x}) = -9.30096 - 0.0047026x_3x_5 - 0.0012547x_1x_3 - 0.0019085x_3x_4 + 20 \leq 0
\end{aligned}$$

where $78 \leq x_1 \leq 102$, $33 \leq x_2 \leq 45$ and $27 \leq x_i \leq 45 (i = 3, 4, 5)$.

Solving the canonical dual problem, we can obtain $\bar{\boldsymbol{\varsigma}} =$

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
\varsigma_1 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 & \varsigma_8 & \varsigma_9 & \varsigma_{10} & \varsigma_{11} \\
336.8388 & 0.0000 & 0.0001 & 0.0002 & 0.0003 & 798.2826 & 2.0310 & 6.1233 & 0.0001 & 1.6054 & 1.1849
\end{array}\right)$$

In this case, $G(\bar{\boldsymbol{\varsigma}}) \succ 0$ and $\text{cond}(G(\bar{\boldsymbol{\varsigma}})) = 9.7330e5$, satisfying the global optimality condition, so we can get $\bar{\boldsymbol{x}} =$

$$\left(\begin{array}{c|c|c|c|c}
x_1 & x_2 & x_3 & x_4 & x_5 \\
77.9452 & 33.0179 & 29.7345 & 44.9884 & 38.2523
\end{array}\right)$$

Noting that the condition number is large, according to the KKT conditions, we can first get

$$\left(\begin{array}{c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & x_5 \\ 78 & 33 & - & 45 & - \end{array}\right).$$

Considering that constraints g_1, g_6 are active, solving the corresponding linear equations, we can easily get $x_3 = 29.995256025681599$, $x_5 = 36.775812905788207$. Finally, the global solution to g04 is $\boldsymbol{x}^* =$

$$\left(\begin{array}{c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & x_5 \\ 78 & 33 & 29.995256025681599 & 45 & 36.775812905788207 \end{array}\right)$$

and $f(\boldsymbol{x}^*) = -3.0666e4$.

Remark 1 We adopt the inverse of $G(\bar{\boldsymbol{\varsigma}})$ because only its smallest eigenvalue approximates to zero although its condition number is large. As a matter of fact, the solution $\bar{\boldsymbol{x}}$ causes only little infeasibility of the first constraint. By integrating the canonical dual solutions with the KKT conditions, we see that x_1, x_2 and x_4 are determined in the first stage.

Example 3 g07

$$\begin{aligned}
\min f(\boldsymbol{x}) = {} & x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3 - 10)^2 + 4(x_4 - 5)^2 + (x_5 - 3)^2 \\
& + 2(x_6 - 1)^2 + 5x_7^2 + 7(x_8 - 11)^2 + 2(x_9 - 10)^2 + (x_{10} - 7)^2 + 45 \\
\text{s.t.} \quad & g_1(\boldsymbol{x}) = -105 + 4x_1 + 5x_2 - 3x_7 + 9x_8 \le 0 \\
& g_2(\boldsymbol{x}) = 10x_1 - 8x_2 - 17x_7 + 2x_8 \le 0 \\
& g_3(\boldsymbol{x}) = -8x_1 + 2x_2 + 5x_9 - 2x_{10} - 12 \le 0 \\
& g_4(\boldsymbol{x}) = 3(x_1 - 2)^2 + 4(x_2 - 3)^2 + 2x_3^2 - 7x_4 - 120 \le 0 \\
& g_5(\boldsymbol{x}) = 5x_1^2 + 8x_2 + (x_3 - 6)^2 - 2x_4 - 40 \le 0 \\
& g_6(\boldsymbol{x}) = x_1^2 + 2(x_2 - 2)^2 - 2x_1x_2 + 14x_5 - 6x_6 \le 0 \\
& g_7(\boldsymbol{x}) = 0.5(x_1 - 8)^2 + 2(x_2 - 4)^2 + 3x_5^2 - x_6 - 30 \le 0 \\
& g_8(\boldsymbol{x}) = -3x_1 + 6x_2 + 12(x_9 - 8)^2 - 7x_{10} \le 0
\end{aligned}$$

where $-10 \le x_i \le 10 (i = 1, \cdots, 10)$.

Solving the canonical dual problem, we can obtain $\bar{\boldsymbol{\varsigma}} =$

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c} \varsigma_1 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 & \varsigma_8 & \varsigma_9 \\ 1.7168 & 0.4746 & 1.3760 & 0.0205 & 0.3120 & 0.2871 & 0.0000 & 0.0000 & 0.0000 \\ \varsigma_{10} & \varsigma_{11} & \varsigma_{12} & \varsigma_{13} & \varsigma_{14} & \varsigma_{15} & \varsigma_{16} & \varsigma_{17} & \varsigma_{18} \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{array}\right)$$

In this case, $G(\bar{\varsigma}) \succ 0$ and $\text{cond}(G(\bar{\varsigma})) = 7.0000$, satisfying the global optimality condition, so we can get $\boldsymbol{x}^* =$

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 & x_{10} \\ 2.1721 & 2.3636 & 8.7746 & 5.0959 & 0.9903 & 1.4307 & 1.3218 & 9.8286 & 8.2800 & 8.3760 \end{array}\right)$$

and $f(\boldsymbol{x}^*) = 24.3111$. Note that there exists little infeasibility due to numerical precision.

Example 4 g10

$$\begin{aligned} \min f(\boldsymbol{x}) &= x_1 + x_2 + x_3 \\ \text{s.t.}\ \ g_1(\boldsymbol{x}) &= -1 + 0.0025(x_4 + x_6) \le 0 \\ g_2(\boldsymbol{x}) &= -1 + 0.0025(x_5 + x_7 - x_4) \le 0 \\ g_3(\boldsymbol{x}) &= -1 + 0.01(x_8 - x_5) \le 0 \\ g_4(\boldsymbol{x}) &= -x_1x_6 + 833.33252x_4 + 100x_1 - 83333.333 \le 0 \\ g_5(\boldsymbol{x}) &= -x_2x_7 + 1250x_5 + x_2x_4 - 1250x_4 \le 0 \\ g_6(\boldsymbol{x}) &= -x_3x_8 + 1250000 + x_3x_5 - 2500x_5 \le 0 \end{aligned}$$

where $100 \le x_1 \le 10000$, $1000 \le x_i \le 10000 (i = 2, 3)$ and $10 \le x_i \le 1000 (i = 4, \cdots, 8)$

Solving the canonical dual problem, we can obtain $\bar{\varsigma} =$

$$\left(\begin{array}{c|c|c|c|c|c|c} \varsigma_1 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 \\ 9.2834 & 28.9205 & 5.8893 & 0.0001 & 0.0001 & 0.0001 & 0.0001 \\ \varsigma_8 & \varsigma_9 & \varsigma_{10} & \varsigma_{11} & \varsigma_{12} & \varsigma_{13} & \varsigma_{14} \\ 0.0001 & 0.0001 & 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{array}\right)$$

In this case, $G(\bar{\varsigma}) \succ 0$ and $\text{cond}(G(\bar{\varsigma})) = 749.4514$, satisfying the global optimality condition. However, the $\max(\text{eig}(G(\bar{\varsigma}))) = 2.5743\text{e}{-4}$, which is too small, so we cannot use the inverse of $G(\bar{\varsigma})$ directly. By the KKT condition, we can find that constraints g_1, g_2, g_3 are active, and all of the box constraints are inactive. That is to say, the problem is equivalent to a linear programming problem with linear constraints, which indicates that g_4, g_5, g_6 must be active. Fixing x_4, x_5, we have

$$\begin{cases} x_1 = \dfrac{83333.333 - 833.33252x_4}{x_4 - 300} \\ x_2 = \dfrac{1250x_4 - 1250x_5}{x_5 - 400} \\ x_3 = 12500 - 25x_5 \\ x_6 = 400 - x_4 \\ x_7 = 400 + x_4 - x_5 \\ x_8 = 100 + x_5 \end{cases}$$

As a result, we can reduce the problem to

$$\min f(\boldsymbol{x}) = \frac{83333.333 - 833.33252x_4}{x_4 - 300} + \frac{1250x_4 - 1250x_5}{x_5 - 400} + 12500 - 25x_5$$

Taking the box constraints of $x_1, \cdots, x_8$ into consideration, when using (100, 200) as an initial point for the unconstrained optimization problem with two variables, it is easy to get the unique minimum $x_4 = 182.0176995811199$ and $x_5 = 295.6011732779338$. Utilizing the equations obtained by the complementary slackness, finally, we have $x_1 = 579.3066844253549$, $x_2 = 1359.970668051655$, $x_3 = 5109.970668051655$, $x_6 = 217.9823004188801$, $x_7 = 286.4165263031861$, $x_8 = 395.6011732779338$ and $f(\boldsymbol{x}^*) = 7049.248020528666$.

Remark 2 We don't use the inverse of $G(\bar{\boldsymbol{\varsigma}})$ directly because all of its eigenvalues are approximately zeros. And the reason why we still use the canonical dual solutions as useful heuristics is that the $G(\bar{\boldsymbol{\varsigma}})$ is slightly positive definite due to the perturbed complementary slackness caused by the SeDuMi. Since all of the box constraints are inactive and the target function is linear, it is not difficult to see that all of the constraints must be active. Note that the constraints of x_4 and x_5 are changed when solving the unconstrained optimization problem since box constraints of x_1, x_2, x_3 and x_6, x_7, x_8 must be satisfied.

Example 5 g18

$$\begin{aligned}
\min f(\boldsymbol{x}) &= -0.5(x_1x_4 - x_2x_3 + x_3x_9 - x_5x_9 + x_5x_8 - x_6x_7) \\
\text{s.t. } & g_1(\boldsymbol{x}) = x_3^2 + x_4^2 - 1 \le 0 \\
& g_2(\boldsymbol{x}) = x_9^2 - 1 \le 0 \\
& g_3(\boldsymbol{x}) = x_5^2 + x_6^2 - 1 \le 0 \\
& g_4(\boldsymbol{x}) = x_1^2 + (x_2 - x_9)^2 - 1 \le 0 \\
& g_5(\boldsymbol{x}) = (x_1 - x_5)^2 + (x_2 - x_6)^2 - 1 \le 0 \\
& g_6(\boldsymbol{x}) = (x_1 - x_7)^2 + (x_2 - x_8)^2 - 1 \le 0 \\
& g_7(\boldsymbol{x}) = (x_3 - x_5)^2 + (x_4 - x_6)^2 - 1 \le 0 \\
& g_8(\boldsymbol{x}) = (x_3 - x_7)^2 + (x_4 - x_8)^2 - 1 \le 0 \\
& g_9(\boldsymbol{x}) = x_7^2 + (x_8 - x_9)^2 - 1 \le 0 \\
& g_{10}(\boldsymbol{x}) = x_2x_3 - x_1x_4 \le 0 \\
& g_{11}(\boldsymbol{x}) = -x_3x_9 \le 0 \\
& g_{12}(\boldsymbol{x}) = x_5x_9 \le 0 \\
& g_{13}(\boldsymbol{x}) = x_6x_7 - x_5x_8 \le 0
\end{aligned}$$

where $-10 \le x_1 \le 10$, $(i = 1, \cdots, 8)$ and $0 \le x_9 \le 20$.

Solving the canonical dual problem, we can obtain $\bar{\boldsymbol{\varsigma}} =$

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c|c|c} \varsigma_1 & \varsigma_2 & \varsigma_3 & \varsigma_4 & \varsigma_5 & \varsigma_6 & \varsigma_7 & \varsigma_8 & \varsigma_9 & \varsigma_{10} & \varsigma_{11} \\ 0.1444 & 0.0000 & 0.1444 & 0.1445 & 0.0000 & 0.1442 & 0.1441 & 0.0000 & 0.1445 & 0.0000 & 0.0000 \\ \varsigma_{12} & \varsigma_{13} & \varsigma_{14} & \varsigma_{15} & \varsigma_{16} & \varsigma_{17} & \varsigma_{18} & \varsigma_{19} & \varsigma_{20} & \varsigma_{21} & \varsigma_{22} \\ 0.0000 & 0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & -0.0000 & 0.0000 \end{array}\right)$$

In this case, $G(\bar{\boldsymbol{\varsigma}}) \succ 0$ and $\text{cond}(G(\bar{\boldsymbol{\varsigma}})) = 7.1887e7$, satisfying the global optimality condition. However, the condition number is large. Taking the KKT conditions into account, we can see that constraints $g_1, g_3, g_4, g_6, g_7, g_9$ are active since the corresponding $\varsigma_1, \varsigma_3, \varsigma_4, \varsigma_6, \varsigma_7, \varsigma_9$ are not zeros. But it becomes still difficult to solve the nonlinear equations. Considering that several eigenvalues of $G(\bar{\boldsymbol{\varsigma}})$ are zeros and there exists no linear term in the objective function, and in this situation, we add a small linear perturbation term $0.05(x_1 + \cdots, x_9)$ to the primal objective function. Solving the perturbed canonical dual problem, we get $G(\bar{\boldsymbol{\varsigma}}) \succ 0$ and $\text{cond}(G(\bar{\boldsymbol{\varsigma}})) = 1.4592e3$ and the smallest eigenvalue of $G(\bar{\boldsymbol{\varsigma}})$ is 0.0021. Therefore, we can get $\bar{\boldsymbol{x}} =$

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ -0.9660 & -0.2585 & -0.2587 & -0.9660 & -0.9661 & -0.2588 & -0.2589 & -0.9657 & 0.0005 \end{array}\right)$$

and $f(\bar{\boldsymbol{x}}) = -0.8663$. Note that there exists little infeasibility due to numerical precision.

Remark 3 The solution we get is quite different from the best known solution. The linear perturbation technique can help to find one of the approximate solutions. As a matter of fact, the following solutions $\bar{\boldsymbol{x}} =$

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ 0.0450 & -0.0387 & 0.8663 & -0.4999 & 0.0004 & -1.0001 & 0.8878 & 0.5000 & 0.9604 \end{array}\right),$$

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ 0.0689 & -0.9972 & 0.9088 & -0.4179 & 0.0920 & -0.9959 & 0.8986 & -0.4388 & 0.0009 \end{array}\right),$$

and

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 & x_9 \\ 0.6888 & -0.7257 & 0.9693 & 0.2454 & 0.6973 & -0.7173 & 0.9726 & 0.2332 & -0.0006 \end{array}\right)$$

can all be considered as approximate solutions, which are obtained by the proposed techniques.

Example 6

$$\begin{aligned} \min f(\boldsymbol{x}) &= -\frac{1}{2}\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} x_i \\ \text{s.t.} \quad g(\boldsymbol{x}) &= \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} x_i \le 0 \end{aligned}$$

where $-1 \le x_i \le 1$, $(i = 1, \cdots, n)$.

Table 1 Comparable running time (s)

n	YALMIP	Proposed method
2	1.0937	0.3080
5	1.3306	0.3131
10	1.6286	0.3234
20	2.4788	0.3392
50	6.2334	0.4749
100	16.1496	0.6542
200	43.1086	2.9885
300	85.8071	8.4530
400	164.9477	10.8982
500	243.2736	16.8634

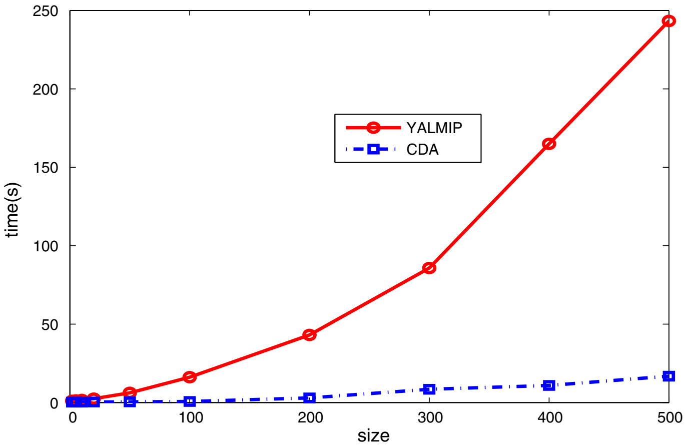

Fig. 1 Illustration of running time for YALMIP and CDA, respectively

When using the proposed techniques to solve the problem, we can get the canonical dual solution $\boldsymbol{\varsigma}^* = (1, \underbrace{0, \cdots, 0}_{n})$ and the corresponding global optimal solution $\boldsymbol{x}^* = (\underbrace{0, \cdots, 0}_{n})$ to the primal problem. Under the same environment, the same problem is solved via a branch and bound method embedded in YALMIP [17]. The running time for two methods is given in Table 1 and Fig. 1, and we can find that the canonical dual algorithm (CDA) consumes much less time.

Example 7 Considering the following special nonconvex QCQP problem

$$\begin{aligned} \min P(\boldsymbol{x}) &= \frac{1}{2}\boldsymbol{x}^T A\boldsymbol{x} - \boldsymbol{a}^T\boldsymbol{x} \\ \text{s.t.} \;\; x_i^2 &= 1, i = 1, \cdots, n \end{aligned}$$

case 1:

$$A = \begin{pmatrix} 0 & 2 & 5 & 7 & 9 & 6 & 4 & 1 & 1 & 3 & 1 & 5 & 3 \\ 2 & 0 & 4 & 6 & 9 & 6 & 5 & 2 & 3 & 4 & 3 & 5 & 4 \\ 5 & 4 & 0 & 2 & 7 & 4 & 4 & 5 & 6 & 8 & 6 & 3 & 5 \\ 7 & 6 & 2 & 0 & 5 & 3 & 4 & 6 & 7 & 10 & 7 & 2 & 5 \\ 9 & 9 & 7 & 5 & 0 & 3 & 5 & 8 & 9 & 11 & 9 & 4 & 6 \\ 6 & 6 & 4 & 3 & 3 & 0 & 2 & 5 & 6 & 8 & 6 & 1 & 3 \\ 4 & 5 & 4 & 4 & 5 & 2 & 0 & 3 & 4 & 7 & 4 & 2 & 1 \\ 1 & 2 & 5 & 6 & 8 & 5 & 3 & 0 & 1 & 4 & 1 & 4 & 2 \\ 1 & 3 & 6 & 7 & 9 & 6 & 4 & 1 & 0 & 2 & 0 & 6 & 3 \\ 3 & 4 & 8 & 10 & 11 & 8 & 7 & 4 & 2 & 0 & 3 & 8 & 5 \\ 1 & 3 & 6 & 7 & 9 & 6 & 4 & 1 & 0 & 3 & 0 & 5 & 3 \\ 5 & 5 & 3 & 2 & 4 & 1 & 2 & 4 & 6 & 8 & 5 & 0 & 3 \\ 3 & 4 & 5 & 5 & 6 & 3 & 1 & 2 & 3 & 5 & 3 & 3 & 0 \end{pmatrix} \quad \boldsymbol{a} = \begin{pmatrix} -4 \\ -4 \\ -2 \\ -3 \\ -6 \\ -2 \\ -2 \\ -3 \\ -5 \\ -7 \\ -5 \\ -2 \\ -3 \end{pmatrix}$$

By solving the corresponding SDP problem in 0.323206 s, we got $\bar{\boldsymbol{\varsigma}}$ = (29.0003, 21.0000, 16.9996, 29.0000, 30.9997, 24.9999, 8.9996, 23.0001, 33.0005, 38.0007, 31.0004, 21.9998, 5.9996), and then $\bar{\boldsymbol{x}} = G^{-1}(\bar{\boldsymbol{\varsigma}})F(\bar{\boldsymbol{\varsigma}})$ = $(-1, -1, 1, 1, 1, 1, 1, -1, -1, -1, -1, 1, -1)$. The same problem was solved by the BARON global optimization package [24] via the MATLAB/BARON Interface Version: v1.57. while the total time elapsed is 0.37 s. We can find that the proposed approach is comparable to BARON for this case.

case 2:

The A and $\boldsymbol{a}$ are given in the next page. By solving the corresponding SDP problem in 0.445124 s, we got $\bar{\boldsymbol{\varsigma}}$ = 1.0e3(1.4337, 1.4377, 0.9288, 1.0418, 1.1448, 0.8929, 0.7829, 0.5839, 0.4999, 0.1261, 0.7631, 0.8151, 1.5443, 1.9394, 2.0274, 1.7933, 1.5722, 1.8804, 1.7274, 1.3933, 1.1102, 1.0551, 1.0851, 0.3251, 0.0842, 0.1171, 0.2331, 0.3131, 0.1692, 0.3809, 0.6019, 0.6488, 0.6388, 0.9558, 1.1188, 1.2357, 1.3648, 1.4567, 1.4767, 1.3828, 1.4767) and then $\bar{\boldsymbol{x}} = G^{-1}(\bar{\boldsymbol{\varsigma}})F(\bar{\boldsymbol{\varsigma}})$ = (1, 1, 1, 1, 1, 1, 1, 1, 1, $-1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1, -1$, 1, 1, 1, 1 1, 1, 1, 1, 1, 1, 1, 1). By using BARON, the same problem is consuming 16.72 s in total. We can find that in this case, the proposed approach is much superior.

$$A = \begin{pmatrix}
0 & 8 & 39 & 37 & 50 & 61 & 58 & 59 & 62 & 81 & 103 & 108 & 145 & 181 & 187 & 161 & 142 & 174 & 185 & 164 & 137 & 117 & 114 & 85 & 77 & 87 & 91 & 105 & 111 & 91 & 83 & 89 & 95 & 74 & 67 & 74 & 57 & 45 & 35 & 29 & 3 \\
8 & 0 & 45 & 47 & 49 & 62 & 60 & 60 & 66 & 81 & 107 & 117 & 149 & 185 & 191 & 170 & 146 & 178 & 186 & 165 & 139 & 122 & 118 & 89 & 80 & 89 & 93 & 106 & 113 & 92 & 85 & 91 & 97 & 81 & 69 & 76 & 59 & 46 & 37 & 33 & 11 \\
39 & 45 & 0 & 9 & 21 & 21 & 16 & 15 & 20 & 40 & 62 & 66 & 104 & 140 & 146 & 120 & 101 & 133 & 142 & 120 & 94 & 77 & 73 & 44 & 36 & 44 & 48 & 62 & 69 & 50 & 42 & 55 & 64 & 44 & 42 & 61 & 46 & 41 & 35 & 30 & 41 \\
37 & 47 & 9 & 0 & 15 & 20 & 17 & 20 & 25 & 44 & 67 & 71 & 108 & 144 & 150 & 124 & 104 & 138 & 143 & 123 & 96 & 80 & 78 & 48 & 40 & 46 & 50 & 63 & 71 & 51 & 43 & 55 & 63 & 43 & 41 & 60 & 41 & 34 & 26 & 21 & 37 \\
50 & 49 & 21 & 15 & 0 & 17 & 18 & 26 & 31 & 50 & 72 & 77 & 114 & 150 & 156 & 130 & 111 & 143 & 140 & 124 & 94 & 83 & 84 & 53 & 46 & 46 & 48 & 64 & 66 & 46 & 38 & 50 & 56 & 35 & 31 & 42 & 25 & 20 & 18 & 18 & 47 \\
61 & 62 & 21 & 20 & 17 & 0 & 6 & 17 & 22 & 41 & 63 & 68 & 106 & 142 & 142 & 115 & 97 & 129 & 130 & 106 & 80 & 68 & 69 & 41 & 34 & 30 & 34 & 47 & 51 & 30 & 22 & 34 & 42 & 23 & 25 & 44 & 30 & 34 & 34 & 35 & 57 \\
58 & 60 & 16 & 17 & 18 & 6 & 0 & 10 & 15 & 35 & 57 & 61 & 99 & 135 & 137 & 110 & 91 & 123 & 126 & 106 & 78 & 62 & 63 & 34 & 27 & 28 & 32 & 46 & 53 & 34 & 26 & 39 & 49 & 30 & 32 & 51 & 36 & 38 & 36 & 33 & 55 \\
59 & 60 & 15 & 20 & 26 & 17 & 10 & 0 & 5 & 24 & 46 & 51 & 88 & 124 & 130 & 104 & 85 & 117 & 124 & 105 & 77 & 60 & 57 & 28 & 19 & 29 & 33 & 49 & 56 & 38 & 32 & 44 & 56 & 39 & 41 & 60 & 47 & 48 & 46 & 40 & 58 \\
62 & 66 & 20 & 25 & 31 & 22 & 15 & 5 & 0 & 20 & 41 & 46 & 84 & 120 & 125 & 105 & 86 & 118 & 128 & 110 & 84 & 61 & 59 & 29 & 21 & 32 & 36 & 54 & 61 & 43 & 36 & 49 & 60 & 44 & 46 & 66 & 52 & 53 & 51 & 45 & 63 \\
81 & 81 & 40 & 44 & 50 & 41 & 35 & 24 & 20 & 0 & 23 & 26 & 63 & 99 & 105 & 90 & 75 & 107 & 118 & 104 & 77 & 50 & 48 & 22 & 14 & 27 & 30 & 48 & 57 & 49 & 51 & 63 & 75 & 62 & 64 & 83 & 71 & 73 & 70 & 65 & 83 \\
103 & 107 & 62 & 67 & 72 & 63 & 57 & 46 & 41 & 23 & 0 & 11 & 49 & 85 & 90 & 72 & 51 & 83 & 93 & 86 & 56 & 34 & 28 & 23 & 29 & 36 & 34 & 46 & 59 & 60 & 63 & 76 & 86 & 78 & 83 & 102 & 93 & 96 & 93 & 87 & 105 \\
108 & 117 & 66 & 71 & 77 & 68 & 61 & 51 & 46 & 26 & 11 & 0 & 40 & 76 & 81 & 62 & 59 & 84 & 101 & 97 & 64 & 42 & 36 & 35 & 40 & 47 & 45 & 59 & 71 & 71 & 75 & 87 & 97 & 89 & 90 & 110 & 98 & 99 & 97 & 91 & 109 \\
145 & 149 & 104 & 108 & 114 & 106 & 99 & 88 & 84 & 63 & 49 & 40 & 0 & 35 & 41 & 34 & 29 & 54 & 72 & 71 & 65 & 49 & 43 & 69 & 77 & 78 & 77 & 85 & 96 & 103 & 106 & 120 & 126 & 121 & 130 & 147 & 136 & 137 & 134 & 117 & 147 \\
181 & 185 & 140 & 144 & 150 & 142 & 135 & 124 & 120 & 99 & 85 & 76 & 35 & 0 & 10 & 31 & 53 & 46 & 69 & 93 & 90 & 82 & 77 & 105 & 114 & 116 & 115 & 119 & 130 & 141 & 142 & 155 & 160 & 159 & 164 & 185 & 172 & 176 & 171 & 166 & 186 \\
187 & 191 & 146 & 150 & 156 & 142 & 137 & 130 & 125 & 105 & 90 & 81 & 41 & 10 & 0 & 27 & 48 & 35 & 58 & 82 & 87 & 77 & 72 & 102 & 111 & 112 & 110 & 115 & 126 & 136 & 140 & 150 & 155 & 155 & 160 & 179 & 172 & 178 & 176 & 171 & 188 \\
161 & 170 & 120 & 124 & 130 & 115 & 110 & 104 & 105 & 90 & 72 & 62 & 34 & 31 & 27 & 0 & 21 & 26 & 58 & 62 & 58 & 60 & 45 & 74 & 84 & 84 & 83 & 88 & 98 & 109 & 112 & 123 & 128 & 127 & 133 & 155 & 148 & 151 & 151 & 144 & 164 \\
142 & 146 & 101 & 104 & 111 & 97 & 91 & 85 & 86 & 75 & 51 & 59 & 29 & 53 & 48 & 21 & 0 & 31 & 43 & 42 & 36 & 30 & 27 & 56 & 64 & 66 & 63 & 66 & 75 & 90 & 93 & 100 & 104 & 108 & 114 & 133 & 126 & 131 & 129 & 125 & 144 \\
174 & 178 & 133 & 138 & 143 & 129 & 123 & 117 & 118 & 107 & 83 & 84 & 54 & 46 & 35 & 26 & 31 & 0 & 26 & 45 & 68 & 62 & 59 & 88 & 96 & 98 & 97 & 98 & 98 & 115 & 126 & 123 & 128 & 136 & 146 & 159 & 158 & 163 & 161 & 157 & 176 \\
185 & 186 & 142 & 143 & 140 & 130 & 126 & 124 & 128 & 118 & 93 & 101 & 72 & 69 & 58 & 58 & 43 & 26 & 0 & 22 & 50 & 70 & 69 & 99 & 107 & 95 & 91 & 79 & 85 & 99 & 108 & 109 & 113 & 124 & 134 & 146 & 147 & 159 & 163 & 156 & 182 \\
164 & 165 & 120 & 123 & 124 & 106 & 106 & 105 & 110 & 104 & 86 & 97 & 71 & 93 & 82 & 62 & 42 & 45 & 22 & 0 & 30 & 49 & 55 & 81 & 87 & 75 & 72 & 59 & 62 & 81 & 88 & 86 & 90 & 101 & 111 & 122 & 124 & 135 & 139 & 139 & 161 \\
137 & 139 & 94 & 96 & 94 & 80 & 78 & 77 & 84 & 77 & 56 & 64 & 65 & 90 & 87 & 58 & 36 & 68 & 50 & 30 & 0 & 21 & 27 & 54 & 60 & 47 & 44 & 31 & 38 & 53 & 60 & 62 & 67 & 75 & 85 & 98 & 121 & 108 & 118 & 113 & 134 \\
117 & 122 & 77 & 80 & 83 & 68 & 62 & 60 & 61 & 50 & 34 & 42 & 49 & 82 & 77 & 60 & 30 & 62 & 70 & 49 & 21 & 0 & 5 & 32 & 40 & 36 & 32 & 36 & 47 & 61 & 64 & 71 & 76 & 79 & 84 & 105 & 97 & 102 & 102 & 95 & 119 \\
114 & 118 & 73 & 78 & 84 & 69 & 63 & 57 & 59 & 48 & 28 & 36 & 43 & 77 & 72 & 45 & 27 & 59 & 69 & 55 & 27 & 5 & 0 & 29 & 37 & 39 & 36 & 42 & 53 & 62 & 66 & 78 & 82 & 81 & 86 & 107 & 99 & 103 & 101 & 97 & 116 \\
85 & 89 & 44 & 48 & 53 & 41 & 34 & 28 & 29 & 22 & 23 & 35 & 69 & 105 & 102 & 74 & 56 & 88 & 99 & 81 & 54 & 32 & 29 & 0 & 8 & 12 & 9 & 28 & 39 & 36 & 39 & 52 & 62 & 54 & 59 & 79 & 71 & 73 & 71 & 67 & 86 \\
77 & 80 & 36 & 40 & 46 & 34 & 27 & 19 & 21 & 14 & 29 & 40 & 77 & 114 & 111 & 84 & 64 & 96 & 107 & 87 & 60 & 40 & 37 & 8 & 0 & 11 & 15 & 33 & 42 & 34 & 36 & 49 & 59 & 50 & 52 & 71 & 65 & 67 & 65 & 60 & 78 \\
87 & 89 & 44 & 46 & 46 & 30 & 28 & 29 & 32 & 27 & 36 & 47 & 78 & 116 & 112 & 84 & 66 & 98 & 95 & 75 & 47 & 36 & 39 & 12 & 11 & 0 & 3 & 21 & 29 & 24 & 27 & 39 & 49 & 42 & 47 & 66 & 59 & 64 & 65 & 62 & 84 \\
91 & 93 & 48 & 50 & 48 & 34 & 32 & 33 & 36 & 30 & 34 & 45 & 77 & 115 & 110 & 83 & 63 & 97 & 91 & 72 & 44 & 32 & 36 & 9 & 15 & 3 & 0 & 20 & 30 & 28 & 31 & 44 & 53 & 46 & 51 & 70 & 63 & 69 & 70 & 67 & 88 \\
105 & 106 & 62 & 63 & 64 & 47 & 46 & 49 & 54 & 48 & 46 & 59 & 85 & 119 & 115 & 88 & 66 & 98 & 79 & 59 & 31 & 36 & 42 & 28 & 33 & 21 & 20 & 0 & 12 & 20 & 28 & 35 & 40 & 43 & 53 & 70 & 67 & 75 & 84 & 79 & 101 \\
111 & 113 & 69 & 71 & 66 & 51 & 53 & 56 & 61 & 57 & 59 & 71 & 96 & 130 & 126 & 98 & 75 & 98 & 85 & 62 & 38 & 47 & 53 & 39 & 42 & 29 & 30 & 12 & 0 & 20 & 28 & 24 & 29 & 39 & 49 & 60 & 62 & 72 & 78 & 82 & 108 \\
91 & 92 & 50 & 51 & 46 & 30 & 34 & 38 & 43 & 49 & 60 & 71 & 103 & 141 & 136 & 109 & 90 & 115 & 99 & 81 & 53 & 61 & 62 & 36 & 34 & 24 & 28 & 20 & 20 & 0 & 8 & 15 & 25 & 23 & 32 & 48 & 46 & 54 & 58 & 62 & 88 \\
83 & 85 & 42 & 43 & 38 & 22 & 26 & 32 & 36 & 51 & 63 & 75 & 106 & 142 & 140 & 112 & 93 & 126 & 108 & 88 & 60 & 64 & 66 & 39 & 36 & 27 & 31 & 28 & 28 & 8 & 0 & 12 & 23 & 14 & 24 & 40 & 38 & 46 & 50 & 53 & 80 \\
89 & 91 & 55 & 55 & 50 & 34 & 39 & 44 & 49 & 63 & 76 & 87 & 120 & 155 & 150 & 123 & 100 & 123 & 109 & 86 & 62 & 71 & 78 & 52 & 49 & 39 & 44 & 35 & 24 & 15 & 12 & 0 & 11 & 14 & 24 & 36 & 37 & 49 & 56 & 59 & 86 \\
95 & 97 & 64 & 63 & 56 & 42 & 49 & 56 & 60 & 75 & 86 & 97 & 126 & 160 & 155 & 128 & 104 & 128 & 113 & 90 & 67 & 76 & 82 & 62 & 59 & 49 & 53 & 40 & 29 & 25 & 23 & 11 & 0 & 21 & 30 & 33 & 43 & 54 & 62 & 66 & 92 \\
74 & 81 & 44 & 43 & 35 & 23 & 30 & 39 & 44 & 62 & 78 & 89 & 121 & 159 & 155 & 127 & 108 & 136 & 124 & 101 & 75 & 79 & 81 & 54 & 50 & 42 & 46 & 43 & 39 & 23 & 14 & 14 & 21 & 0 & 9 & 25 & 23 & 34 & 41 & 45 & 71 \\
67 & 69 & 42 & 41 & 31 & 25 & 32 & 41 & 46 & 64 & 83 & 90 & 130 & 164 & 160 & 133 & 114 & 146 & 134 & 111 & 85 & 84 & 86 & 59 & 52 & 47 & 51 & 53 & 49 & 32 & 24 & 24 & 30 & 9 & 0 & 18 & 13 & 24 & 32 & 38 & 64 \\
74 & 76 & 61 & 60 & 42 & 44 & 51 & 60 & 66 & 83 & 102 & 110 & 147 & 185 & 179 & 155 & 133 & 159 & 146 & 122 & 98 & 105 & 107 & 79 & 71 & 66 & 70 & 70 & 60 & 48 & 40 & 36 & 33 & 25 & 18 & 0 & 17 & 29 & 38 & 45 & 71 \\
57 & 59 & 46 & 41 & 25 & 30 & 36 & 47 & 52 & 71 & 93 & 98 & 136 & 172 & 172 & 148 & 126 & 158 & 147 & 124 & 121 & 97 & 99 & 71 & 65 & 59 & 63 & 67 & 62 & 46 & 38 & 37 & 43 & 23 & 13 & 17 & 0 & 12 & 21 & 27 & 54 \\
45 & 46 & 41 & 34 & 20 & 34 & 38 & 48 & 53 & 73 & 96 & 99 & 137 & 176 & 178 & 151 & 131 & 163 & 159 & 135 & 108 & 102 & 103 & 73 & 67 & 64 & 69 & 75 & 72 & 54 & 46 & 49 & 54 & 34 & 24 & 29 & 12 & 0 & 9 & 15 & 41 \\
35 & 37 & 35 & 26 & 18 & 34 & 36 & 46 & 51 & 70 & 93 & 97 & 134 & 171 & 176 & 151 & 129 & 161 & 163 & 139 & 118 & 102 & 101 & 71 & 65 & 65 & 70 & 84 & 78 & 58 & 50 & 56 & 62 & 41 & 32 & 38 & 21 & 9 & 0 & 6 & 32 \\
29 & 33 & 30 & 21 & 18 & 35 & 33 & 40 & 45 & 65 & 87 & 91 & 117 & 166 & 171 & 144 & 125 & 157 & 156 & 139 & 113 & 95 & 97 & 67 & 60 & 62 & 67 & 79 & 82 & 62 & 53 & 59 & 66 & 45 & 38 & 45 & 27 & 15 & 6 & 0 & 25 \\
3 & 11 & 41 & 37 & 47 & 57 & 55 & 58 & 63 & 83 & 105 & 109 & 147 & 186 & 188 & 164 & 144 & 176 & 182 & 161 & 134 & 119 & 116 & 86 & 78 & 84 & 88 & 101 & 108 & 88 & 80 & 86 & 92 & 71 & 64 & 71 & 54 & 41 & 32 & 25 & 0
\end{pmatrix}$$

$$\boldsymbol{a} = -\begin{pmatrix} 5 & 12 & 55 & 41 & 53 & 64 & 61 & 61 & 66 & 84 & 111 & 113 & 150 & 186 & 192 & 166 & 147 & 180 & 188 & 167 & 140 & 124 & 119 & 90 & 87 & 90 & 94 & 107 & 114 & 77 & 86 & 92 & 98 & 80 & 74 & 77 & 60 & 48 & 38 & 32 & 6 \end{pmatrix}^T$$

5 Conclusion

Canonical duality theory was applied to solve a class of CEC benchmark constrained optimization problems. Experimental results showed that some of them can be solved directly, some of them can be solved by integrating the canonical dual solutions with the KKT conditions, and others can be solved approximately by adding a small linear perturbation term. Additional special examples demonstrated the superiority of the proposed approach.

Acknowledgments We would like to thank Professor Nick Sahinidis very much for providing us a free license for using the MATLAB/BARON interface. And we would also like to thank two anonymous reviewers and the handling editor, whose suggestions greatly helped to improve the paper's presentation. This paper was partially supported by a Grant (AFOSR FA9550-10-1-0487) from the US Air Force Office of Scientific Research. Dr. Xiaojun Zhou's work was supported by the China Scholarship Council and Dr. Chunhua Yang's work wassupported by the National Science Found for Distinguished Young Scholars of China (Grant No. 61025015) and the Foundation for Innovative Research Groups of the National Natural Science Foundation of China (Grant No. 61321003).

References

1. Anstreicher, K.M.: Semidefinite programming versus the reformulation-linearization technique for nonconvex quadratically constrained quadratic programming. J. Glob. Optim. **43**, 471–484 (2009)
2. Audet, C., Hansen, P., Jaumard, B., Savard, G.: A branch and cut algorithm for nonconvex quadratically constrained quadratic programming. Math. Progr. **87**(1), 131–152 (2000)
3. Bao, X.W., Sahinidis, N.V., Tawarmalani, M.: Semidefinite relaxations for quadratically constrained quadratic programming: a review and comparisons. Math. Progr. **129**(1), 129–157 (2011)
4. Bazaraa, M.S., Sherali, H.D., Shetty, C.M.: Nonlinear Programmming: Theory and Algorithms, 3rd edn. Wiley, New York (2006)
5. Boyd, S., Vandenberghe, L.: Convex Optimization. Cambridge University Press, Cambridge (2004)
6. Coello, C.A.C.: Theoretical and numerical constraint-handling techniques used with evolutionary algorithms: a survey of the state of the art. Comput. Methods Appl. Mech. Eng. **191**, 1245–1287 (2002)
7. Cottle, R.W.: Manifestations of Schur complement. Linear Algebra Appl. **8**, 189–211 (1974)
8. Fang, S.C., Gao, D.Y., Sheu, R.L., Wu, S.Y.: Canonical dual approach for solving 0–1 quadratic programming problems. J. Ind. Manag. Optim. **4**, 125–142 (2008)
9. Gao, D.Y., Strang, G.: Geometric nonlinearity: potential energy, complementary energy, and the gap function. Q. J. Appl. Math. **XLVII**(3), 487–504 (1989)
10. Gao, D.Y.: Canonical duality theory and solutions to constrained nonconvex quadratic programming. J. Glob. Optim. **29**(4), 377–399 (2004)
11. Gao, D.Y., Ruan, N., Sherali, H.D.: Solutions and optimality criteria for nonconvex constrained global optimization problems with connections between canonical and Lagrangian duality. J. Glob. Optim. **45**, 473–497 (2009)
12. Gao, D.Y.: Caonical duality theory: Unified understanding and generalized solution for global optimization problems. Comp. Chem. Eng. **33**, 1964–1972 (2009)
13. Gao, D.Y., Sherali, H.D.: Canonical duality: Connection between nonconvex mechanics and global optimization. In: Advances in Applied Mathematics and Global Optimization, pp. 257–326, Springer (2009)
14. Gao, D.Y., Ruan, N.: Solutions to quadratic minimization problems with box and integer constraints. J. Glob. Optim. **47**, 463–484 (2010)
15. Liang, J.J., Runarsson, T.P., Mezura-Montes, E., Clerc, M., Suganthan, P.N., Coello, C.A.C., Deb, K.: Problem definitions and evaluation criteria for the CEC 2006 special session on constrained real-parameter optimization, Tech. Rep, Nanyang Technological University, Singapore (2006)
16. Linderoth, J.: A simplicial branch-and-bound algorithm for solving quadratically constrained quadratic programs. Math. Progr. **103**(2), 251–282 (2005)

17. Löfberg, J.: YALMIP : a toolbox for modeling and optimization in MATLAB. In: IEEE International Symposium on Computer Aided Control Systems Design, pp. 284–289. Taipei (2004)
18. Lu, C., Fang, S.C., Jin, Q.W., Wang, Z.B., Xing, W.X.: KKT solution and conic relaxation for solving quadratically constrained quadratic programming problems. SIAM J. Optim. **21**(4), 1475–1490 (2011)
19. Michalewicz, Z.: A survey of constraint handling techniques in evolutionary computation methods. In: McDonnell, J.R., Reynolds, R.G., Fogel, D.B. (eds.) Proceedings of the Fourth Annual Conference on Evolutionary Programming, pp. 135–155. MIT Press, Cambridge (1995)
20. Misener, R., Floudas, C.A.: Global optimization of mixed-integer quadratically-constrained quadratic programs (MIQCQP) through piecewise-linear and edge-concave relaxations. Math. Progr. **136**(1), 155–182 (2012)
21. Ruan, N., Gao, D.Y.: Canonical duality approach for non-linear dynamical systems, to appear in IMA. J. Appl. Math. doi:10.1093/imamat/hxs067
22. Ruan, N., Gao, D.Y.: Global optimal solutions to a general sensor network localization problem. Perform Eval **75**, 1–16 (2013)
23. Sturn, J.F.: Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones. Optim. Methods Softw. **11**, 625–653 (1999)
24. Tawarmalani, M., Sahinidis, N.V.: A polyhedral branch-and-cut approach to global optimization. Math. Progr. **103**(2), 225–249 (2005)
25. Wang, Z.B., Fang, S.C., Gao, D.Y., Xing, W.X.: Canonical dual approach to solving the maximum cut problem. J. Glob. Optim. **54**, 341–352 (2012)
26. Zhou, X.J., Yang, C.H., Gui, W.H.: State transition algorithm. J. Ind. Manag. Optim. **8**(4), 1039–1056 (2012)
27. Zhou, X.J., Yang, C.H., Gui, W.H.: Nonlinear system identification and control using state transition algorithm. Appl. Math. Comput. **226**, 169–179 (2014)